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DYNAMIC AXISYMMETRIC PROBLEM OF THE DIRECT PIEZOEFFECT FOR A RADIALLY POLARIZED ANISOTROPIC PIEZOCERAMIC CYLINDER

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An unsteady problem of electroelasticity for a long radially polarized anisotropic piezoceramic cylinder with normal stresses that are arbitrary functions of time applied to the radial surfaces is considered. A closed solution is constructed by the method of expansion with respect to eigen vector-functions. This solution allows determining the frequencies of natural oscillations, the stress–strain state of the element, and the potential and intensity of the induced electric field.

Key words: *coupled problem of direct piezoeffect, long cylinder, axisymmetric dynamic load.*

Introduction. In studying dynamic problems of electroelasticity for a radially polarized anisotropic hollow cylinder, it is necessary to integrate a complicated system of partial differential equations. Therefore, solutions of such problems were mainly obtained in the steady regime of forced oscillations by approximate methods [1] or by the method of expansion with respect to eigen vector-functions for some one-dimensional elements [2, 3]. In the case of the above-mentioned polarization, exact solutions for a finite inhomogeneous cylinder were constructed in eigenvalue problems [2, 3].

In the present work, we construct an exact closed solution of the electroelasticity problem for a long piezoceramic anisotropic cylinder under an arbitrary unsteady axisymmetric action.

1. Formulation of the Problem. Let a long hollow cylinder, which is a linearly elastic anisotropic solid, be made of a piezoceramic material with induced polarization along the radius r_*. We consider the case with a dynamic load (normal stresses) $q_1^*(t_*)$ and $q_2^*(t_*)$ applied to electroded non-fixed curvilinear surfaces $r_* = a$ and $r_* = b$. The outer radial surface is connected to a measurement instrument with a large input resistance, which corresponds to the "idle" regime (absence of free electric charges), while the inner surface is grounded.

The boundary-value problem considered in this formulation simulates the operation of piezoelectric elements in instruments with the direct piezoeffect, which transform mechanical actions to the corresponding electric signal.

The system of differential equations and the boundary and initial conditions of the dynamic problem of the elasticity theory considered here have the following form in dimensionless variables [1, 2]:

$$\nabla^2 U - \frac{C_{11}}{C_{33}}\frac{U}{r^2} + e_{33}\nabla^2\varphi - e_{31}\frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{\partial^2 U}{\partial t^2} = 0,$$
$$C_{33}\varepsilon_{33}\nabla^2\varphi - e_{33}\nabla^2 U - e_{31}\frac{1}{r}\frac{\partial U}{\partial r} = 0; \qquad (1)$$

for $r = 1, k$,

$$\sigma_{rr}\Big|_{r=1} = \frac{\partial U}{\partial r} + \frac{C_{13}}{C_{33}}U + e_{33}\frac{\partial\varphi}{\partial r} = q_1(t), \qquad \sigma_{rr}\Big|_{r=k} = \frac{\partial U}{\partial r} + \frac{C_{13}}{C_{33}}\frac{U}{k} + e_{33}\frac{\partial\varphi}{\partial r} = q_2(t),$$
$$D_r\Big|_{r=1} = -C_{33}\varepsilon_{33}\frac{\partial\varphi}{\partial r} + e_{31}U + e_{33}\frac{\partial U}{\partial r} = 0, \qquad \varphi(k,t) = 0; \qquad (2)$$

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for $t = 0$,

$$U(r,0) = U_0(r), \qquad \dot U(r,0) = \dot U_0(r). \tag{3}$$

Here, $U = U^*/b$, $r = r_*/b$, $k = a/b$, $q_1(t) = q_1^*/C_{33}$, $q_2(t) = q_2^*/C_{33}$, $t = t_* b^{-1}\sqrt{C_{33}/\rho}$, $\varphi = \varphi^*/(bC_{33})$, $\sigma_{rr}(r_*, t_*)$, $D_r(r_*, t_*)$, and $U^*(r_*, t_*)$ are the radial components of the tensor of mechanical stresses, electric induction, and displacement vector, respectively, $\varphi^*(r_*, t_*)$ is the electric field potential, ρ, C_{ms}, e_{ms} ($m = \overline{1,3}$, $s = \overline{1,3}$), and ε_{33} are the volume density, elastic constants, piezomoduli, and dielectric permeability of the anisotropic electroelastic material, U_0 and $\dot U_0$ are the displacements and displacement rates known at the initial time, and $\nabla^2 = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r$; the dot means differentiation with respect to t.

Relations (1)–(3) are the mathematical formulation of the initial-boundary problem of electroelasticity considered.

2. Construction of the General Solution. The electroelasticity boundary-value problem (1)–(3) is solved by the method of finite integral transformations (FITs) [4] with respect to the radial coordinate r.

The boundary-value problem (1)–(3) is brought to a standard form (with homogeneous boundary conditions on the curvilinear surfaces of the cylinder) on the basis of the presentation

$$U(r,t) = H_1(r,t) + u(r,t), \qquad \varphi(r,t) = H_2(r,t) + \chi(r,t), \tag{4}$$

where $H_1(r,t) = f_1(r)q_1(t) + f_2(r)q_2(t)$ and $H_2(r,t) = f_3(r)q_1(t) + f_4(r)q_2(t)$.

Substituting Eqs. (4) into system (1)–(3) with allowance for the conditions

$$\begin{gathered} f_1(1) = f_1(k) = f_2(1) = f_2(k) = f_1'(k) = f_2'(1) = f_3(k) = f_3'(k) = f_4(1) = f_4'(1) = 0, \\ f_1'(1) + e_{33}f_3'(1) = 1, \qquad e_{33}f_1'(1) - C_{33}\varepsilon_{33}f_3'(1) = 0, \\ f_2'(k) + e_{33}f_4'(k) = 1, \qquad e_{33}f_2'(k) - C_{33}\varepsilon_{33}f_4'(k) = 0, \end{gathered} \tag{5}$$

we obtain the following initial-boundary problem with respect to $u(r,t)$ and $\chi(r,t)$:

$$\begin{gathered} \nabla^2 u - \frac{C_{11}}{C_{33}}\,\frac{u}{r^2} + e_{33}\nabla^2\chi - e_{31}\,\frac{1}{r}\,\frac{\partial\chi}{\partial r} - \frac{\partial^2 u}{\partial t^2} = B_1, \\ C_{33}\varepsilon_{33}\nabla^2\chi - e_{33}\nabla^2 u - e_{31}\,\frac{1}{r}\,\frac{\partial u}{\partial r} = B_2; \end{gathered} \tag{6}$$

for $r = 1, k$,

$$\begin{gathered} \frac{\partial u}{\partial r} + \frac{C_{13}}{C_{33}}\,\frac{u}{r} + e_{33}\,\frac{\partial\chi}{\partial r} = 0, \\ \Big(-C_{33}\varepsilon_{33}\,\frac{\partial\chi}{\partial r} + e_{31}u + e_{33}\,\frac{\partial u}{\partial r}\Big)\Big|_{r=1} = 0, \qquad \chi(k,t) = 0; \end{gathered} \tag{7}$$

for $t = 0$,

$$\begin{gathered} u_0(r) = U_0(r) - H_1(r,0), \qquad \dot u_0(r) = \dot U_0(r) - \dot H_1(r,0), \\ B_1 = -\nabla^2 H_1 + \frac{C_{11}}{C_{33}}\,\frac{H_1}{r^2} - e_{33}\nabla^2 H_2 + e_{31}\,\frac{1}{r}\,\frac{\partial H_2}{\partial r} + \frac{\partial^2 H_1}{\partial t^2}, \\ B_2 = -C_{33}\varepsilon_{33}\nabla^2 H_2 + e_{33}\nabla^2 H_1 + e_{31}\,\frac{1}{r}\,\frac{\partial H_1}{\partial r}. \end{gathered} \tag{8}$$

The prime here means differentiation with respect to r.

The functions $f_1(r), \ldots, f_4(r)$ are found from the equations

$$f_1^{\rm IV}(r) = f_2^{\rm IV}(r) = 0, \qquad f_3'''(r) = f_4'''(r) = 0. \tag{9}$$

On the segment $[k, 1]$, we introduce a degenerate FIT [4] with unknown components of the vector-function of the transformation kernel $K_1(\lambda_i, r)$ and $K_2(\lambda_i, r)$:

$$G(\lambda_i, t) = \int_k^1 u(r,t) K_1(\lambda_i, r) r\, dr; \qquad (10)$$

$$u(r,t) = \sum_{i=1}^{\infty} G(\lambda_i, t) K_1(\lambda_i, r) \|K_i\|^{-2}, \qquad \chi(r,t) = \sum_{i=1}^{\infty} G(\lambda_i, t) K_2(\lambda_i, r) \|K_i\|^{-2}. \qquad (11)$$

Here,

$$\|K_i\|^2 = \int_k^1 K_1^2(\lambda_i, r) r\, dr,$$

λ_i ($i = \overline{1, \infty}$) are the positive parameters that form a countable set. The circular frequencies of axisymmetric oscillations of the cylinder ω_i are related to λ_i as

$$\omega_i = \frac{\lambda_i}{b} \sqrt{\frac{C_{33}}{\rho}}.$$

Subjecting system (6) and the initial conditions (8) to FIT (10), in accordance with the algorithm developed, we obtain a countable set of the Cauchy problems for the transformant $G(\lambda_i, t)$, whose solution has the form

$$G(\lambda_i, t) = G_0(\lambda_i) \cos \lambda_i t + \dot{G}_0(\lambda_i) \lambda_i^{-1} \sin \lambda_i t - \lambda_i^{-1} \int_0^t F(\lambda_i, \tau) \sin \lambda_i (t - \tau)\, d\tau,$$

and, taking into account the boundary conditions (7), a homogeneous boundary-value problem for the FIT kernel components $K_1(\lambda_i, r)$ and $K_2(\lambda_i, r)$:

$$\begin{gathered} \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\Big) K_1 - \frac{C_{11}}{C_{33}} \frac{K_1}{r^2} + \lambda_{in}^2 K_1 + e_{33} \frac{d^2 K_2}{dr^2} + (e_{33} - e_{31}) \frac{1}{r} \frac{dK_2}{dr} = 0, \\ e_{33} \frac{d^2 K_1}{dr^2} + (e_{33} + e_{31}) \frac{1}{r} \frac{dK_1}{dr} - C_{33}\varepsilon_{33} \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\Big) K_2 = 0; \end{gathered} \qquad (12)$$

for $r = k, 1$,

$$\Big(K_1' + \frac{C_{13}}{C_{33}} \frac{K_1}{r} + e_{33} K_2'\Big)\Big|_{r=k} = 0, \qquad (-C_{33}\varepsilon_{33} K_2' + e_{31} K_2 + e_{33} K_1')\Big|_{r=1} = 0,$$

$$K_2(\lambda_i, k) = 0, \qquad (13)$$

$$F(\lambda_i, t) = \int_k^1 (B_1 K_1 + B_2 K_2) r\, dr, \quad G_0(\lambda_i) = \int_k^1 u_0 K_1 r\, dr, \quad \dot{G}_0(\lambda_i) = \int_k^1 \dot{u}_0 K_1 r\, dr.$$

The system of differential equations (12) is reduced to the following resolving equation of the third order with respect to the new function $K_3(\lambda_i, r)$:

$$\Big(\frac{d}{dr} + \frac{1}{r}\Big)\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \beta_i^2 - \frac{\nu^2}{r^2}\Big) K_3(\lambda_i, r) = 0. \qquad (14)$$

Here,

$$K_3 = \frac{dK_2}{dr} + A \frac{dK_1}{dr} + \frac{C_{11}}{C_{33} e_{31}} \frac{K_1}{r}, \qquad \beta_i^2 = \frac{C_{33}\varepsilon_{33}\lambda_i^2}{e_{33}^2 - e_{33} C_{33}\varepsilon_{33} A},$$

$$\nu^2 = \Big[e_{31} - e_{33} + C_{33}\varepsilon_{33}\Big(A + \frac{C_{11}}{C_{33} e_{31}}\Big)\Big] \Big[(e_{33} - C_{33}\varepsilon_{33} A)^{-1} - \frac{e_{31}}{e_{33}^2 + C_{33}\varepsilon_{33}}\Big],$$

$$A = -\frac{e_{31} e_{33} + \sqrt{(e_{31} e_{33})^2 + C_{11}\varepsilon_{33}(e_{33}^2 + C_{33}\varepsilon_{33}) + C_{33}\varepsilon_{33} e_{31}^2}}{C_{33}\varepsilon_{33} e_{31}}.$$

The general solution of Eq. (14) is written in the form [5]

$$K_3(\lambda_i, r) = C_{1i} J_\nu(\beta_i r) + C_{2i} Y_\nu(\beta_i r) + C_{3i} E_\nu(\beta_i r),$$

where $J_\nu(\,\cdot\,)$, $Y_\nu(\,\cdot\,)$, and $E_\nu(\,\cdot\,)$ are the Bessel functions of the first and second kind and the Lommer–Weber of the order ν.

In the case considered, the function $E_\nu(\beta_i r)$ is determined by the equality

$$E_\nu(\beta_i r) = \sum_{k=0,2,4,\ldots}^{\infty} a_k (\beta_i r)^{k+1} \qquad \Big(a_0 = \beta_i, \qquad a_k = -\frac{a_{k-2}}{(k+1)^2 - \nu^2}\Big).$$

Taking into account the dependences between $K_1(\lambda_i, r)$, $K_2(\lambda_i, r)$, and $K_3(\lambda_i, r)$ obtained by reducing system (12) to Eq. (14) and the recurrent relations for the Bessel functions [5], we obtain the following expressions for the FIT kernel components:

$$K_1(\lambda_i, r) = \sum_{j=1}^{3} C_{ji} N_j(\lambda_i, r), \qquad K_2(\lambda_i, r) = \sum_{j=1}^{3} C_{ji} W_j(\lambda_i, r) + C_{4i}. \tag{15}$$

Here,

$$N_1(\lambda_i, r) = (m_{1i}\nu + m_{2i}) r^{-1} J_\nu(\beta_i r) - m_{1i}\beta J_{\nu+1}(\beta_i r),$$

$$N_2(\lambda_i, r) = (m_{1i}\nu + m_{2i}) r^{-1} Y_\nu(\beta_i r) - m_{1i}\beta Y_{\nu+1}(\beta_i r),$$

$$N_3(\lambda_i, r) = \sum_{k=0,2,4,\ldots}^{\infty} a_k [m_{1i}\beta_i(k+1) + m_{2i}](\beta_i r)^k,$$

$$W_1(\lambda_i, r) = \int \Big[J_\nu(\beta_i r) + \Big(A\nu - \frac{C_{11}}{C_{33} e_{31}}\Big) r^{-1} J_\nu(\beta_i r) - A\beta_i J_{\nu+1}(\beta_i r)\Big]\, dr,$$

$$W_2(\lambda_i, r) = \int \Big[Y_\nu(\beta_i r) + \Big(A\nu - \frac{C_{11}}{C_{33} e_{31}}\Big) r^{-1} Y_\nu(\beta_i r) - A\beta_i Y_{\nu+1}(\beta_i r)\Big]\, dr,$$

$$W_3(\lambda_i, r) = a_0(m_{1i}\beta_i + m_{2i})\Big[A - \frac{C_{11}}{C_{33} e_{31} \beta_i}\,\ln(r)\Big] + \sum_{k=0,2,4,\ldots}^{\infty} a_k [\beta_i (k+2)]^{-1} (\beta_i r)^{k+2}$$

$$+ \sum_{k=2,4,6,\ldots}^{\infty} a_k \Big(A - \frac{C_{11}}{C_{33} e_{31} \beta_i k}\Big)[m_{1i}\beta_i(k+1) + m_{2i}](\beta_i r)^k,$$

$$m_{1i} = -\lambda_i^{-2}\Big(e_{33} + \frac{C_{33}\varepsilon_{33}(1 + e_{33}A)}{e_{33} - C_{33}\varepsilon_{33}A}\Big), \qquad m_{2i} = m_{1i} + e_{31}\lambda_i^{-2}.$$

With allowance for Eqs. (15), the boundary conditions (13) yield a system of homogeneous equations with respect to the constants of integration $C_{1i}, \ldots, C_{4i}$. Finding the nontrivial solution of this system, we obtain a transcendental equation for determining the eigenvalues λ_i and also expressions for the constants $C_{1i}, \ldots, C_{4i}$.

3. Calculation Results. The final stage of the present study is determining the functions $f_1(r), \ldots, f_4(r)$ involved into expansions (4). Using the differential equations (9) and the boundary conditions (5), we obtain

$$f_1(r) = \frac{C_{33}\varepsilon_{33}}{C_{33}\varepsilon_{33} + e_{33}^2}\,[r^3 - (2k+1)r^2 + k(k+2)r - k^2](k-1)^{-2},$$

$$f_2(r) = \frac{C_{33}\varepsilon_{33}}{C_{33}\varepsilon_{33} + e_{33}^2}\,[r^3 - (k+2)r^2 + (2k+1)r - k](k-1)^{-2}, \tag{16}$$

$$f_3(r) = \frac{e_{33}}{2(C_{33}\varepsilon_{33} + e_{33}^2)}\,[r^2 - 2kr + k^2](1-k)^{-1}, \qquad f_4(r) = -f_3(r)\,\frac{r^2 - 2r + 2k - k^2}{r^2 - 2kr + k^2}.$$

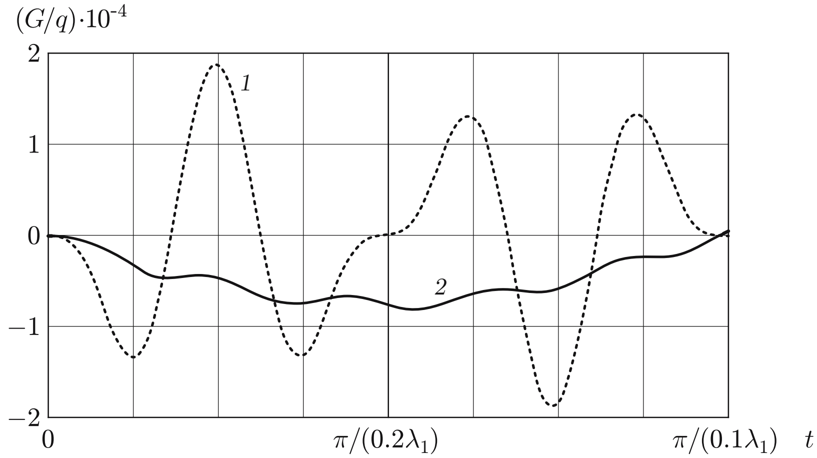

Fig. 1. Time evolution of the transformants $G(\lambda_1, n, t)$ making the main contribution to the stress–strain state of the cylinder ($a/b = 0.2$) for different frequencies of forced oscillations: $\theta = 0.6\lambda_1$ (1) and $\theta = 0.1\lambda_1$ (2).

TABLE 1

Eigenvalues λ_i of Free Axisymmetric Oscillations of the Cylinders

i	Piezoceramic cylinder			Ceramic cylinder		
	$a/b = 0.2$	$a/b = 0.5$	$a/b = 0.8$	$a/b = 0.2$	$a/b = 0.5$	$a/b = 0.8$
1	1.985	1.421	1.139	2.150	1.658	1.333
2	5.334	7.425	14.633	5.358	6.592	15.784
3	9.442	14.294	32.441	8.499	12.720	31.570

Applying the FIT inversion formulas (11) to transformant (10) and taking into account Eq. (4), we obtain expressions for determining the displacements $U(r, t)$ and the electric field potential $\varphi(r, t)$:

$$U(r,t) = f_1(r)q_1(t) + f_2(r)q_2(t) + \sum_{i=1}^{\infty} G(\lambda_i, t)K_1(\lambda_i, r)\|K_i\|^{-2},$$

$$\varphi(r,t) = f_3(r)q_1(t) + f_4(r)q_2(t) + \sum_{i=1}^{\infty} G(\lambda_i, t)K_2(\lambda_i, r)\|K_i\|^{-2}. \quad (17)$$

Functions (17) satisfy the differential equations (1), the boundary conditions (2), and the initial conditions (3), i.e., they are a closed solution of the electroelasticity problem considered.

The difference of the potentials $V(t)$ between the electroded radial planes of the piezoceramic cylinder is determined by the formula [6]

$$V(t) = \varphi(1, t) - \varphi(k, t).$$

Taking into account Eqs. (13), (16), and (17), we finally obtain

$$V(t) = \frac{e_{33}(1-k)}{2(C_{33}\varepsilon_{33} + e_{33}^2)}\,[q_1(t) + q_2(t)] + \sum_{i=1}^{\infty} G(\lambda_i, t)K_2(\lambda_i, 1)\,\|K_i\|^{-2}.$$

4. Analysis of Numerical Results. As an example, let us consider a piezoceramic cylinder (made of the TsTS-19 composition) with the radial surfaces subjected to the load

$$q_1(t) = q\sin\theta t, \qquad q_2(t) = 0$$

(θ is the frequency of forced oscillations).

The calculations were performed for the following initial data [2]: $e_{31} = -4.9$ C/m^2, $e_{33} = 14.9$ C/m^2, $C_{11} = 10.9 \cdot 10^{10}$ N/m^2, $C_{13} = 5.4 \cdot 10^{10}$ N/m^2, $C_{33} = 9.3 \cdot 10^{10}$ N/m^2, and $\varepsilon_{33} = 7.26 \cdot 10^{-9}$ F/m.

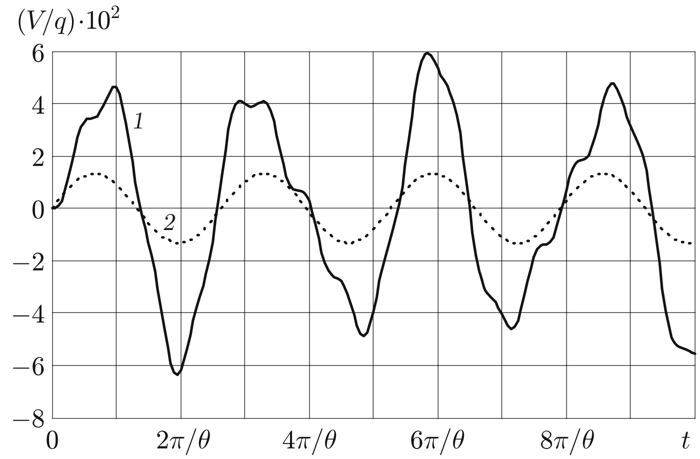

Fig. 2. Difference of potentials between the electroded radial surfaces of the piezoceramic cylinder versus time (1) and oscillogram of the external load (2) for $a/b = 0.2$ and $\theta = 0.6\lambda_1$.

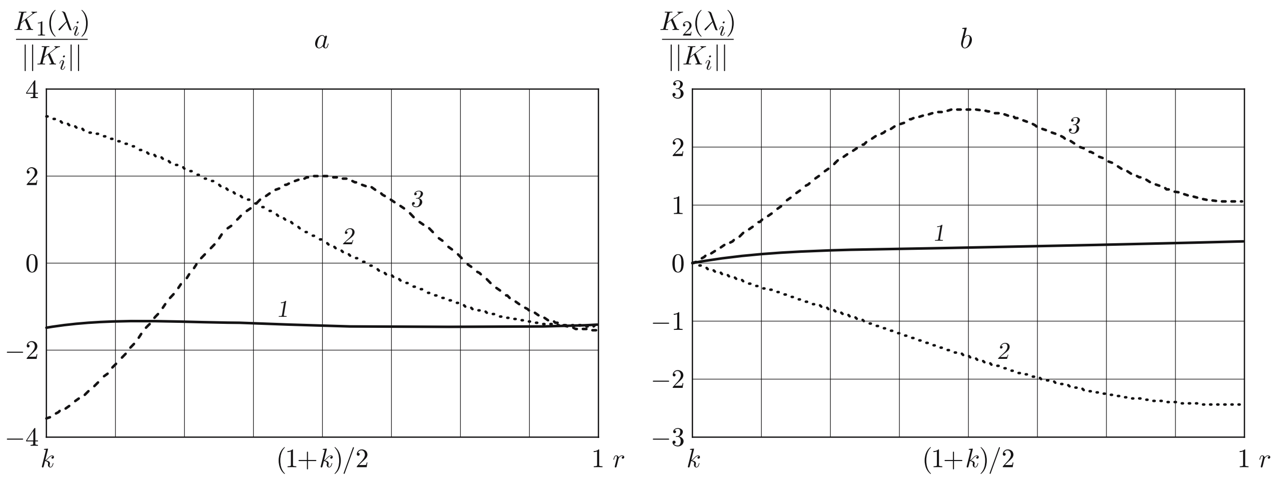

Fig. 3. Normalized shapes $K_1(\lambda_i, r)$ (a) and $K_2(\lambda_i, r)$ (b) of free oscillations of the cylinder for $a/b = 0.2$ and $i = 1$ (1), 2 (2), and 3 (3).

The eigenvalues λ_i of free axisymmetric oscillations of elements made of piezoceramic and ceramic materials with identical elastic characteristics are listed in Table 1. The calculations were performed for different values of the relative thickness a/b. An analysis of results shows that taking into account the coupling between the mechanical and electric fields of stresses exerts a significant effect on the value of ω_i. An increase in the parameter a/b leads to a decrease in the parameter λ_1. In comparing the frequencies of the next tones of oscillations, however, we see the opposite dependence; moreover, the frequency spectrum becomes less dense.

Figure 1 shows the time evolution of the transformants $G(\lambda_1, n, t)$ making the main contribution to the stress–strain state of the cylinder ($a/b = 0.2$) for different frequencies of forced oscillations.

The calculated results show that the assumption about the steady character of the regime of forced oscillations used in studying dynamic problems is valid only if the frequencies of natural and forced oscillations are substantially different. Under a high-frequency external action, the character of variation of the stress state of the system in time is more complicated because of superposition of the reflected strain waves. To confirm this conclusion, Fig. 2 shows the dependence $V(t)$ for $a/b = 0.2$ and $\theta = 0.6\lambda_1$.

Figure 3 shows the normalized shapes $K_1(\lambda_i, r)$ and $K_2(\lambda_i, r)$ of free oscillations of the cylinder for $a/b = 0.2$.

It should be noted that the character of the dependence $K_1(\lambda_i, r)$ in the piezoceramic cylinder, as well as in a standard anisotropic element, is mainly determined by the number of zeroes in this dependence. The zero value of the function $K_2(\lambda_i, r)$ at $r = k$ corresponds to electric grounding of the inner radial surface.

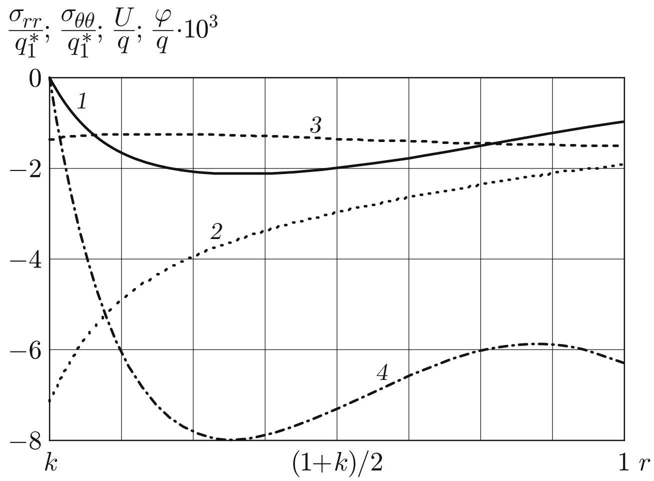

Fig. 4. Amplitude values of mechanical and electric functions over the thickness for $a/b = 0.2$ and $\theta = 0.2\lambda_1$: σ_{rr} (1), $\sigma_{\theta\theta}$ (2), U (3), and φ (4).

Figure 4 shows the behavior of the amplitude values of mechanical and electric functions over the thickness for $a/b = 0.2$ and $\theta = 0.2\lambda_1$.

On one hand, the results obtained confirm the known fact that the radial component of the normal stresses is substantially smaller than the circumferential component; on the other hand, these data do not validate the results of numerical calculations [2], which predict a linear character of the variation of the stress tensor components. Note that it is only the radial component of the displacement vector that changes almost linearly; the remaining functions follow complicated curvilinear dependences.

Finally, we should note that the algorithm of the solution constructed also makes it possible to consider the "short-circuit" case and to obtain results that can be used for determining the electroacoustic sensitivity of converters.

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